

NOTE

AN INFINITE FAMILY OF INTEGRAL GRAPHS

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We obtain here an infinite family of integral complete tripartite graphs.

The purpose of this note is to obtain an infinite family of integral complete tripartite graphs. For background see [1].

We recall first some definitions and facts. A complete n -partite graph $K(p_1, \dots, p_n)$ is a graph with a set $A = A_1 \cup \dots \cup A_n$ of $p_1 + \dots + p_n$ vertices, where A_i are nonempty disjoint sets, $|A_i| = p_i$ for $1 \leq i \leq n$, such that two vertices in A are adjacent if and only if they belong to different A_i, A_j . A graph is called *integral* if all the eigenvalues of its adjacency matrix are integers.

The eigenvalues of a complete n -partite graph $K(p_1, \dots, p_n)$ with $p_1 < p_2 < \dots < p_n$ are exactly the roots of the equation $\sum_{i=1}^n p_i/(x + p_i) = 1$.

We obtain here an infinite number of triples (p_1, p_2, p_3) of different positive integers such that all the roots of the equation $\sum_{i=1}^3 p_i/(x + p_i) = 1$ are integral.

Define

$$\begin{aligned} p_1 &= 4u^2(u^2 + v^2)^3, & p_2 &= 4v^2(u^2 + v^2)^3, \\ p_3 &= 3u^2v^2(u^2 + 6uv + v^2)(-u^2 + 6uv - v^2) \\ &= 3u^2v^2(34u^2v^2 - u^4 - v^4), \end{aligned}$$

where u, v are positive integers such that $(3 - \sqrt{8})v < u < v$.

For example, for $u = 1, v = 2$, we get $p_1 = 500, p_2 = 2000, p_3 = 1428$.

We show first that p_1, p_2, p_3 are positive integers different from each other. Indeed, the condition $(3 - \sqrt{8})v < u < v$ ensures $-u^2 + 6uv - v^2 > 0$ and so $p_3 > 0$. We have $p_1 < p_2$ because $0 < u < v$. Assume $p_1 = p_3$ and let $u = tu_0, v = tv_0$, where $(u_0, v_0) = 1$. Then,

$$4(u_0^2 + v_0^2)^3 = 3u_0^2v_0^2(34u_0^2v_0^2 - u_0^4 - v_0^4)$$

and so

$$u_0^2 + v_0^2 \equiv 0 \pmod{3} \Rightarrow u_0 \equiv v_0 \equiv 0 \pmod{3},$$

contradicting the fact $(u_0, v_0) = 1$. Therefore $p_1 \neq p_3$. Similarly $p_2 \neq p_3$.

We get by direct computation that the roots of the equation $\sum_{i=1}^3 p_i/(x+p_i) = 1$ are the following:

$$\begin{aligned}x_1 &= 24u^2v^2(u^2+v^2)^2, \\x_2 &= -2uv(u^2+v^2)^2(u^2+6uv+v^2), \\x_3 &= -2uv(u^2+v^2)^2(-u^2+6uv-v^2).\end{aligned}$$

Remark. It is enough to verify that x_1 and x_2 are roots of the polynomial

$$\begin{aligned}f(x) &= \prod_{i=1}^3 (x-p_i) - p_1(x+p_2)(x+p_3) \\&\quad - p_2(x+p_1)(x+p_3) - p_3(x+p_1)(x+p_2),\end{aligned}$$

because $-p_1, -p_2, -p_3$ are not roots of $f(x)$, the sum of the roots of $f(x)$ is 0 and $x_1 + x_2 + x_3 = 0$, $x_1 \neq x_2$.

We show now how the formulas above were obtained. Let us be given the equation

$$\sum_{i=1}^3 \frac{p_i}{x+p_i} = 1, \quad (1)$$

where $1 \leq p_1 < p_2 < p_3$ are integers.

The rational roots of (1) are necessarily integers, because they are also the roots of the monic polynomial $f(x)$ defined above. Let x_1 be a root of (1). Clearly $x_1 \neq 0$. Let $\alpha_i = x_1/p_i$ ($1 \leq i \leq 3$), so $\sum_{i=1}^3 1/(1+\alpha_i) = 1$. Any root of (1) different from x_1 is of the form xx_1 ($x \neq 1$), where x satisfies the equation $\sum_{i=1}^3 1/(1+x\alpha_i) = 1$. Therefore

$$0 = \sum_{i=1}^3 \frac{1}{1+x\alpha_i} - \sum_{i=1}^3 \frac{1}{1+\alpha_i} = \sum_{i=1}^3 \frac{\alpha_i(1-x)}{(1+x\alpha_i)(1+\alpha_i)},$$

and so

$$\sum_{i=1}^3 \frac{\alpha_i}{(1+x\alpha_i)(1+\alpha_i)} = 0.$$

We have

$$\begin{aligned}&\frac{\alpha_1}{1+\alpha_1} (1+x\alpha_2)(1+x\alpha_3) + \frac{\alpha_2}{1+\alpha_2} (1+x\alpha_1)(1+x\alpha_3) \\&\quad + \frac{\alpha_3}{1+\alpha_3} (1+x\alpha_1)(1+x\alpha_2) = 0.\end{aligned}$$

We get

$$\begin{aligned}&x^2\alpha_1\alpha_2\alpha_3 \sum_{i=1}^3 \frac{1}{1+\alpha_i} + x \left[\frac{\alpha_1}{1+\alpha_1} (\alpha_2+\alpha_3) + \frac{\alpha_2}{1+\alpha_2} (\alpha_1+\alpha_3) + \frac{\alpha_3}{1+\alpha_3} (\alpha_1+\alpha_2) \right] \\&\quad + \frac{\alpha_1}{1+\alpha_1} + \frac{\alpha_2}{1+\alpha_2} + \frac{\alpha_3}{1+\alpha_3} = 0.\end{aligned}$$

As $\sum_{i=1}^3 1/(1+\alpha_i) = 1$, we obtain

$$\begin{aligned} 0 &= \alpha_1 \alpha_2 \alpha_3 x^2 + \left[(\alpha_1 + \alpha_2 + \alpha_3) \sum_{i=1}^3 \frac{\alpha_i}{1+\alpha_i} - \sum_{i=1}^3 \frac{\alpha_i^2}{1+\alpha_i} \right] x + 2 \\ &= \alpha_1 \alpha_2 \alpha_3 x^2 + (\alpha_1 + \alpha_2 + \alpha_3 + 2)x + 2. \end{aligned}$$

The roots of the last equation are rational if and only if its discriminant $(\alpha_1 + \alpha_2 + \alpha_3)^2 - 8\alpha_1 \alpha_2 \alpha_3$ is a square of a rational number. As $\alpha_3 = (\alpha_1 + \alpha_2 + 2)/(\alpha_1 \alpha_2 - 1)$, we obtain the condition

$$\alpha_1 \alpha_2 (\alpha_1 + \alpha_2 + 2)(\alpha_1 \alpha_2 (\alpha_1 + \alpha_2 - 6) + 8) \text{ is a square} \quad (2)$$

Now we look for rationals α_1, α_2 which fulfil condition (2) and also $\alpha_1 + \alpha_2 - 6 = 0$. Condition (2) takes now the form: $\alpha_1(6 - \alpha_1)$ is a square, that is, there exists β rational positive such that $\alpha_1^2 - 6\alpha_1 + \beta^2 = 0$. For given rational β , the roots of the equation $x^2 - 6x + \beta^2$ are rational if and only if $9 - \beta^2$ is a square: $9 = \beta^2 + \gamma^2$ (γ rational positive). Let $\beta = s/r$, $\gamma = t/r$, where s, t, r are positive integers, so $s^2 + t^2 = 9r^2$. We have $s \equiv t \equiv 0 \pmod{3}$, $s = 3s_1$, $t = 3t_1$ so finally we obtain $s_1^2 + t_1^2 = r^2$. Solving this pythagorean equation leads us to the formulas above.

It is probable that for $n > 3$ there also exist an infinite number of complete n -partite graphs with integral spectra.

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Reference

- [1] F. Harary and A.J. Schwenk, Which graphs have integral spectra?, in: *Graphs and Combinatorics*, Lecture Notes in Mathematics 406 (Springer, Berlin, 1974).